

COUNTING POLYNOMIALS OVER FINITE FIELDS WITH GIVEN ROOT MULTIPLICITIES

AYAH ALMOUSA AND MELANIE MATCHETT WOOD

ABSTRACT. We give formulas for the number of polynomials over a finite field with given root multiplicities, in particular in cases when the formula is surprisingly simple (a power of q). Besides this concrete interpretation, we also prove an analogous result on configuration spaces in the Grothendieck ring of varieties, suggesting new homological stabilization conjectures for configuration spaces of the plane.

1. INTRODUCTION

Given a finite field \mathbb{F}_q , a monic polynomial $f \in \mathbb{F}_q[x]$ factors into linear factors $(x - \alpha_1)^{e_1} \cdots (x - \alpha_t)^{e_t}$ over the algebraic closure $\bar{\mathbb{F}}_q$ (with $\alpha_i \in \bar{\mathbb{F}}_q$ distinct). To f , we can associate the partition $P(f) = e_0 \cdots e_t$ (using multiplicative notation for partitions). For a partition λ , we define

$$w_\lambda := \#\{\text{monic } f \in \mathbb{F}_q[x] \mid P(f) = \lambda\}.$$

For example, $w_{12} = q^2 - q$, where the subscript “12” denotes the partition with two elements 1, 2. The number of square-free monic polynomials of degree $n \geq 2$ over \mathbb{F}_q is $w_{1^n} = q^n - q^{n-1}$, a well-known fact.

For two partitions λ, λ' , we define the refinement ordering $\lambda \leq \lambda'$ if λ can be partitioned into subsets that add to the elements of λ' , so for example $11247 \leq 357$ (see Section 2). We define

$$\bar{w}_\lambda = \sum_{\lambda' \geq \lambda} w_{\lambda'},$$

so for example $\bar{w}_{1^n} = q^n$ as $\{\lambda' \geq 1^n\}$ is the set of all partitions of n . Also, $\bar{w}_{1^{n_2}} = \bar{w}_{1^{n+2}} - w_{1^{n+2}} = q^{n+1}$. We have that $\bar{w}_{1^n d}$ counts polynomials with at least one root with multiplicity at least d , and we will see that $\bar{w}_{1^n d} = q^{n+1}$. Similarly, $\bar{w}_{1^n 22}$ counts polynomials with either at least two roots with multiplicity at least 2 or at least one root with multiplicity at least 4, and $\bar{w}_{1^n 22} = q^{n+2}$. As another example $\bar{w}_{12259} = q^5$. These examples, and many more, lead to the natural conjecture that

$$(1) \quad \bar{w}_\lambda \stackrel{?}{=} q^{|\lambda|},$$

for all λ , given for example as a comment to [Ell]. In fact, this conjecture is false as $w_{11223} = q^5 + q^2 - q$, as pointed out in [VW12, Section 2].

In this paper, we address the question of when Equation (1) is true and we prove the following.

Theorem 1.1. *For integers $m \geq -1$, and $k \geq 0$, and $b_i, e_i \geq 1$ for $0 \leq i \leq m$, such that each $1 \leq i \leq m$, we have $b_i \geq \sum_{j < i} e_j b_j$, we have*

$$\bar{w}_{1^k b_0 e_0 \dots b_m e_m} = q^{k + \sum_i e_i}.$$

This theorem proves a large number of cases of when Equation (1) holds, including those mentioned above. In the special cases when $m = 1$ or both $m = 2$ and $e_0 = 1$, Theorem 1.1 follows from [VW12, 5.20]. It is natural to consider the $\overline{w}_{1^k\lambda}$ together for varying k , because they all count polynomials with multiple roots “at least as bad” as λ , as in the examples with $\lambda = d$ and $\lambda = 22$ given above. For any λ , it is the case that the limit

$$(2) \quad \lim_{k \rightarrow \infty} \frac{\overline{w}_{1^k\lambda}}{q^k}$$

exists [VW12, Theorem 1.33], but for the partitions $\lambda = b_0^{e_0} \cdots b_m^{e_m}$ satisfying the hypothesis of Theorem 1.1, we see that in fact $\frac{\overline{w}_{1^k\lambda}}{q^k}$ is independent of k .

To prove Theorem 1.1, we start with the idea of the proof of [VW12, Lemma 5.18] and add new ideas that allow us to extend well beyond the cases of $\lambda = 1^k ab^r$ that [VW12] could prove. In [VW12, 5.20], a stronger version of the theorem is proven, one about classes of configuration spaces in the Grothendieck ring of varieties over any field (see Section 4), and our proof works in that generality as well, resulting in Theorem 4.2.

The limits (2) have analogous limits for classes of configuration spaces in the Grothendieck ring of varieties, which have very interesting connections to the homological stabilization of configuration spaces in topology (see [VW12, 1.41-1.50] for more details). For example, if $\text{Conf}^n X$ is the space of unordered n -tuples of distinct points on a manifold X , then the dimension of the i th rational homology group $h_i(\text{Conf}^n X)$ stabilizes for n sufficiently large (given i), a recent result of Church [Chu12] and Randal-Williams [RW12] for closed manifolds and an older result of McDuff [McD75] for open manifolds. In the case that $X = \mathbb{R}^2$, this homological stability is an even older result of Arnol’d [Arn69], and moreover, Arnol’d shows that the $h_i(\text{Conf}^n X)$ are independent of n for $n \geq 2$. Arnol’d’s result is analogous to (could be predicted by) the fact that w_{1^n} , or equivalently $\overline{w}_{1^{n-2}2}$, is independent of n for $n \geq 2$, and further, the exact values of $h_i(\text{Conf}^n \mathbb{R}^2)$ that Arnol’d gives could be predicted from the exact values of $\overline{w}_{1^{n-2}2}$.

Let $\text{Conf}_c^\lambda(X)$ denote the space of unordered tuples of points of a manifold X whose multiplicity partition λ' satisfies $\lambda' \not\geq \lambda$. Informally, $\text{Conf}_c^\lambda(X)$ is the complement of points with multiplicities that are λ “or worse” (where the “worse” configurations are those in the closure of the configurations with multiplicity λ). For example, $\text{Conf}_c^{1^{n-2}2}(X) = \text{Conf}^n X$, and $\text{Conf}_c^{1^{n-d}d}(X)$ is the space of unordered sets of n points of X in which all points appear with multiplicity at most $d-1$. Theorem 1.1 (and Equation (5)) then motivates the following topological conjecture, extending [VW12, 1.43 Conjecture E].

Conjecture 1.2. *For integers $m, k \geq 0$, and $b_i, e_i, d \geq 1$ for $0 \leq i \leq m$, such that for all i , $b_i \geq \sum_{j < i} e_j b_j$, we have*

$$h_\ell(\text{Conf}_c^{1^k b_0^{e_0} \cdots b_m^{e_m}}(\mathbb{R}^{2d})) = \begin{cases} 1 & \ell = 0 \text{ or } \ell = 2d(\sum_i e_i(b_i - 1)) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In the case when $d = 1$ and $b_0^{e_0} \cdots b_m^{e_m} = 2$, this is Arnol’d’s theorem [Arn69], O. Randall-Williams has shown the conjecture for any d when $b_0^{e_0} \cdots b_m^{e_m} = b_0$, and according to T. Church, Arnol’d’s work [Arn70] can be used to show the conjecture for arbitrary d when $b_0^{e_0} \cdots b_m^{e_m} = b_0^{e_0}$ (see [VW12, Section 1.44]). Our conjecture goes well beyond the cases that are currently known, and the recently proven cases were motivated by [VW12, 1.43 Conjecture E], a special case of our conjecture, made for the same reasons.

1.1. Further directions. It would be interesting to have a complete classification for which λ we have that $\frac{\overline{w}_1^k \lambda}{q^k}$ is independent of k (perhaps for k sufficiently large). Further, we are curious whether the classification is the same as when the dimensions of the homology groups of the analogous configuration spaces are independent of k . We are also particularly curious as to whether there are examples in which $\frac{\overline{w}_1^k \lambda}{q^k}$ is independent of $k \gg 0$ but not a power of q . The question of counting polynomials is the case of counting points on the affine line (which gives $X = \mathbb{A}_{\mathbb{C}}^1 = \mathbb{R}^2$ in the topological analog), and we are curious for what other spaces and partitions λ does counting points with multiplicity λ or worse give this independence in k .

1.2. Outline of the paper. In Section 2 we specify our notation for the paper. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we give the refinement of Theorem 1.1 that we have proven on configuration spaces of any variety in the Grothendieck ring of varieties over any field.

Acknowledgements. The first author was supported by National Science Foundation grant DMS-1147782 and the second author was supported by American Institute of Mathematics Five-Year Fellowship and National Science Foundation grant DMS-1147782.

2. NOTATION

In this paper, a *partition* λ is a multiset, and we use a multiplicative notation so that $\lambda = a_1^{e_1} \cdots a_m^{e_m}$ is the multiset in which a_i occurs e_i times. (We avoid two-digit numbers so that, for example, $\lambda = 12$ is the two element multiset including the elements 1 and 2.) We let $|\lambda| = \sum_i e_i$ be the size of the multiset.

Suppose λ, λ' , and π are partitions. If x, y, z are elements with $x + y = z$ such that $\lambda = xy\pi$ and $\lambda' = z\pi$, we say λ' is an *elementary merge* of λ . In this case $|\lambda| = 1 + |\lambda'|$. We define the *refinement ordering* $<$ on partitions as generated by elementary merges. (If λ' is an elementary merge of λ , then $\lambda < \lambda'$.) For example, $123 < 3^2 < 6$. We write $\lambda \leq \lambda'$ if $\lambda < \lambda'$ or $\lambda = \lambda'$.

If $\lambda = a_1^{e_1} \cdots a_m^{e_m}$ with the a_i distinct, we could (equivalently to the above) define w_λ to be the number of m -tuples (f_1, \dots, f_m) in which f_i is a square-free monic polynomial in $\mathbb{F}_q[x]$ of degree e_i and the f_i are pairwise relatively prime. (We can associate to (f_1, \dots, f_m) the polynomial $\prod_i f_i^{a_i}$ with partition λ .) Again equivalently, we could define w_λ to be the number of assignments to each monic irreducible $f \in \mathbb{F}_q[x]$ an integer n_f between 0 and m , inclusive, so that $\sum_{f \text{ with } n_f=i} \deg(f) = e_i$ for all $i \geq 1$. (We can associate such an assignment to a tuple (f_1, \dots, f_m) with $f_i = \prod_{f \text{ with } n_f=i} f$.) In this way we can define w_λ for the a_i in any set, not just for a_i positive integers. Further, we note that w_λ only depends on the *multiplicity sequence* e_i of λ .

3. PROOF OF THEOREM 1.1

First we need two lemmas.

Lemma 3.1. *If A is a formal variable, we have $\overline{w}_{A^k(bA)} = q^{k+1}$ for all $k \geq 0$ and $b \geq 1$.*

Proof. We have

$$\overline{w}_{A^k(bA)} = \sum_{\lambda \geq A^k(bA)} w_\lambda,$$

and by dividing each element of each λ in the sum by A , we see that $\overline{w}_{A^k(bA)} = \overline{w}_{1^k b}$, which is q^{k+1} [VW12, Proposition 5.9(b)]. We give a proof here for completeness. Let c_n be the number of monic polynomials in $\mathbb{F}_q[x]$ of degree n in which every root appears with multiplicity at most $b-1$, and $d_n = q^n$ be the number of monic polynomials in $\mathbb{F}_q[x]$ of degree n . Since we can factor any monic $f \in \mathbb{F}_q[x]$ uniquely as $g(x)h(x)^b$, so that every root of g has multiplicity at most $b-1$ and $g(x)$ and $h(x)$ are both monic, we have an equality of generating functions

$$(1 - tq)^{-1} = \sum_n d_n t^n = \sum_{n,m} c_m d_n t^{m+bn}.$$

Thus $\sum_m c_m t^m = (1 - t^b q)/(1 - tq)$, and so $c_n = q^n - q^{n-b+1}$ for $n \geq b$. The lemma follows because $\overline{w}_{1^k b} = \overline{w}_{1^{k+b}} - c_{k+b} = q^{k+1}$. \square

Lemma 3.2 (Formal product rule). *Let A, B_0, \dots, B_m be formal variables and b_0, e_0, \dots, e_m be integers at least 1. Then*

$$\overline{w}_{A^k(b_0 A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} = \overline{w}_{A^k(b_0 A)}\overline{w}_{B_0^{e_0-1}}\overline{w}_{B_1^{e_1}}\dots\overline{w}_{B_m^{e_m}}.$$

Proof. We see that each side counts the following: the number of ways to assign to each irreducible monic polynomial $f \in \mathbb{F}_q[x]$ a tuple $(a_f A, b_{f,0} B_0, \dots, b_{f,m} B_m)$ such that 1) $a_f, b_{f,i}$ are non-negative integers, 2) $\sum_f a_f \deg(f) = k+b_0$ and $\sum_f b_{f,0} \deg(f) = e_0-1$ and $\sum_f b_{f,i} \deg(f) = e_i$ for $i > 0$, and 3) at least one a_f is at least b_0 . On the left-hand side of the lemma, such as assignment corresponds to one element counted by w_λ , where λ contains the element $nA + n_0 B_0 + \dots + n_m B_m$ exactly e times, where $e = \sum_f \text{with } a_f=n, b_{f,i}=n_i \text{ for all } i \deg(f)$. The element counted is the tuple composed of the square-free polynomials $\prod_f \text{with } a_f=n, b_{f,i}=n_i \text{ for all } i f$. On the right-hand side of the lemma, such as assignment corresponds to an $m+2$ tuple of elements counted by $w_\lambda, w_{\lambda_0}, \dots, w_{\lambda_m}$, respectively, where, λ contains the element nA exactly e times, where $e = \sum_f \text{with } a_f=n \deg(f)$, and for all i , we have that λ_i contains the element $n_i B_i$ exactly e_i times, where $e_i = \sum_f \text{with } b_{f,i}=n_i \deg(f)$. \square

We fix integers $m \geq -1$, and $k \geq 0$, and $b_i, e_i \geq 1$ for $0 \leq i \leq m$, such that for all i ,

$$(3) \quad b_i \geq \sum_{j < i} e_j b_j.$$

We will now prove Theorem 1.1 by induction on $\sum_i e_i$, where the base case $m = -1$ and $\sum_i e_i = 0$ is clear. If μ is a partition, let \mathcal{R}_μ be the set of partitions $\geq \mu$. The map ϕ that sends $A \mapsto 1$ and $B_i \mapsto b_i$ for formal variables A, B_i induces a map of posets

$$\mathcal{R}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \rightarrow \mathcal{R}_{1^{k+b_0}b_0^{e_0-1}b_1^{e_1}\dots b_m^{e_m}}.$$

We will see that ϕ restricts to a *bijection*

$$(4) \quad \mathcal{R}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \setminus \mathcal{R}_{A^k(b_0 A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \rightarrow \mathcal{R}_{1^{k+b_0}b_0^{e_0-1}b_1^{e_1}\dots b_m^{e_m}} \setminus \mathcal{R}_{1^k b_0^{e_0}b_1^{e_1}\dots b_m^{e_m}},$$

and that this bijection preserves the multiplicity sequence of each partition. We let π_i be the map on integers that is reduction to standard representatives modulo b_i , and note that it induces a map on partitions of integers.

Lemma 3.3. *If $0 \leq r \leq b_0 - 1$, and $0 \leq s_0 \leq e_0 - 1$, and $0 \leq s_i \leq e_i$ for $i \geq 1$, then if we successively apply π_m, \dots, π_l to $\phi(rA + \sum_i s_i B_i)$, the the map π_l reduces $\pi_{l+1} \circ \dots \circ \pi_m \circ \phi(rA + \sum_i s_i B_i) = r + \sum_{i \leq l} s_i b_i$ by exactly $s_l b_l$.*

Proof. We induct downwards on l . We have $r + \sum_{i \leq l-1} s_i b_i \leq b_0 - 1 + (\sum_{i \leq l-1} e_i b_i) - b_0 \leq b_l - 1$ by Equation 3. Since $r + \sum_{i \leq l-1} s_i b_i \geq 0$, it must be that $r + \sum_{i \leq l-1} s_i b_i \geq 0$ is the standard reduction of $r + \sum_{i \leq l} s_i b_i$ modulo b_l . \square

We first see that ϕ restricts to a map as in Equation 4. Suppose for contradiction that for some $\lambda \in \mathcal{R}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \setminus \mathcal{R}_{A^k(b_0A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}}$, we have $\phi(\lambda) = \mu \geq 1^k b_0^{e_0} b_1^{e_1} \dots b_m^{e_m}$. Let the elements of μ be $\mu_j = r_j + \sum_i s_{j,i} b_i$, with r_j and $s_{j,i}$ non-negative integers, not all 0 for a fixed j , such that $\sum_j r_j = k$, and $\sum_j s_{j,i} = e_i$ for all i . We have that $\pi_m(\mu_j)$ reduces μ_j by at least $s_{j,m} b_m$, and since we know by Lemma 3.3 that the total reduction of elements of $\phi(\lambda) = \mu$ is exactly $e_m b_m$, it must be that $\pi_m(\mu_j)$ reduces μ_j by exactly $s_{j,m} b_m$. Similarly, we make the same argument for the successively applied π_{m-1}, \dots, π_0 , but then we have a contradiction as π_0 reduces the elements of $\pi_1 \circ \dots \circ \pi_m(\mu)$ by at least $e_0 b_0$ total, but the elements of $\pi_1 \circ \dots \circ \phi(\lambda)$ by $(e_0 - 1)b_0$ by Lemma 3.3.

Next we see that ϕ gives a bijection in Equation 4. In fact, Lemma 3.3 has the following corollary.

Corollary 3.4. *Let $0 \leq r, r' \leq b_0 - 1$, and $0 \leq s_0, s'_0 \leq e_0 - 1$, and $0 \leq s_i, s'_i \leq e_i$ for $i \geq 1$. If*

$$r + \sum_i s_i b_i = r' + \sum_i s'_i b_i,$$

then $r = r'$ and $s_i = s'_i$ for all i .

Proof. We successively apply π_m, \dots, π_0 to obtain $s_i = s'_i$ by Lemma 3.3, and the final remainder is $r = r'$. \square

So, we see that if $\lambda, \lambda' \in \mathcal{R}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \setminus \mathcal{R}_{A^k(b_0A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}}$, with $\phi(\lambda) = \phi(\lambda')$ then $\lambda = \lambda'$, for if e, e' are elements of λ, λ' respectively, then $\phi(e) = \phi(e')$ implies $e = e'$ by Corollary 3.4.

Finally, Corollary 3.4 implies that ϕ in Equation 4 preserves multiplicity sequences of partitions, as for $\lambda \in \mathcal{R}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \setminus \mathcal{R}_{A^k(b_0A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}}$ the application of ϕ does not make any two unequal elements of λ equal.

Since w_π only depends on the multiplicity sequence of π , we have

$$\begin{aligned} \overline{w}_{1^{k+b_0}b_0^{e_0-1}b_1^{e_1}\dots b_m^{e_m}} - \overline{w}_{1^k b_0^{e_0} b_1^{e_1} \dots b_m^{e_m}} &= \sum_{\lambda \in \mathcal{R}_{1^{k+b_0}b_0^{e_0-1}b_1^{e_1}\dots b_m^{e_m}} \setminus \mathcal{R}_{1^k b_0^{e_0} b_1^{e_1} \dots b_m^{e_m}}} w_\lambda \\ &= \sum_{\mu \in \mathcal{R}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} \setminus \mathcal{R}_{A^k(b_0A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}}} w_\mu \\ &= \overline{w}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} - \overline{w}_{A^k(b_0A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}}. \end{aligned}$$

By Lemmas 3.1 and 3.2, we have that $\overline{w}_{A^{k+b_0}B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} = q^{k+b_0-1+\sum_i e_i}$ and $\overline{w}_{A^k(b_0A)B_0^{e_0-1}B_1^{e_1}\dots B_m^{e_m}} = q^{k+\sum_i e_i}$. By induction, we have that $\overline{w}_{1^{k+b_0}b_0^{e_0-1}b_1^{e_1}\dots b_m^{e_m}} = q^{k+b_0-1+\sum_i e_i}$, and so Theorem 1.1 follows.

4. IN THE GROTHENDIECK RING OF VARIETIES

Let \mathbf{k} be a field. The Grothendieck ring of varieties $\mathcal{M} := K_0(\text{Var}_{\mathbf{k}})$ is defined as follows. As an abelian group, it is generated by the classes of finite type \mathbf{k} -schemes up to isomorphism. The class of a scheme X in \mathcal{M} is denoted $[X]$. The group relations are generated by the following “cut and paste” relations: if Y is a closed subscheme of X , and U is its (open) complement, then $[X] = [U] + [Y]$. The product $[X][Y] := [X \times_{\mathbf{k}} Y]$ makes \mathcal{M} into a commutative ring.

For a partition $\lambda = a_1^{e_1} \cdots a_m^{e_m}$ with a_i distinct, we define $w_\lambda(X)$ to be the open subscheme of $\prod_i \text{Sym}^{e_i} X$ in which all the points are distinct, i.e. the complement of the “big diagonal”. (Note that taking $\mathbf{k} = \mathbb{F}_q$ applying the \mathbb{F}_q -point counting functor to $w_\lambda(\mathbb{A}^1)$ recovers the integer w_λ defined above.) Define $\overline{w}_\lambda(X) = \sum_{\lambda' \geq \lambda} [w_{\lambda'}(X)]$.

Let $\mathbf{Z}_X(t) := \sum_{n \geq 0} [\text{Sym}^n X] t^n \in \mathcal{M}[[t]]$ be the *motivic zeta function* (defined by Kapranov, [Kap00, (1.3)]). If $\mathbf{k} = \mathbb{F}_q$, then the \mathbb{F}_q -point counting functor sends $\mathbf{Z}_X(t)$ to the Weil zeta function $\zeta_X(s)$, where $t = q^{-s}$. For a partition λ , we define $\overline{K}_{X,1 \bullet \lambda}(t) := \sum_j \overline{w}_{1^j \lambda}(X) t^j \in \mathcal{M}[[t]]$. See [VW12, 1.1-1.11 and Section 2] for a more detailed introduction to the above topics in this context.

As the motivic analog of Theorem 1.1, for $\lambda = b_0^{e_0} \cdots b_m^{e_m}$ satisfying the hypotheses of Theorem 1.1, we will determine $\overline{K}_{X,1 \bullet \lambda}(t)$ in terms of $\mathbf{Z}_X(t)$. The following will replace Lemma 3.1.

Lemma 4.1 (Proposition 5.9(b) of [VW12]). *For an integer $a > 1$, we have*

$$\overline{K}_{X,1 \bullet a}(t) = t^{-a} \mathbf{Z}_X(t) (1 - 1/\mathbf{Z}_X(t^a)).$$

Theorem 4.2 (Refinement of Theorem 1.1 in the Grothendieck ring). *For a variety X over \mathbf{k} , and integers $m \geq -1$, and $k \geq 0$, and $b_i, e_i \geq 1$ for $0 \leq i \leq m$, such that each $1 \leq i \leq m$, we have $b_i \geq \sum_{j < i} e_j b_j$, we have*

(1) *for formal variables A, B_i , and $m \geq 0$, we have*

$$\overline{w}_{1^k b_0^{e_0} b_1^{e_1} \cdots b_m^{e_m}}(X) = \overline{w}_{1^{k+b_0} b_0^{e_0-1} b_1^{e_1} \cdots b_m^{e_m}}(X) - \overline{w}_{A^{k+b_0} B_0^{e_0-1} B_1^{e_1} \cdots B_m^{e_m}}(X) + \overline{w}_{A^k (b_0 A) B_0^{e_0-1} B_1^{e_1} \cdots B_m^{e_m}}(X),$$

and

$$\overline{K}_{X,1 \bullet b_0^{e_0} b_1^{e_1} \cdots b_m^{e_m}}(t) = \overline{K}_{X,1 \bullet b_0^{e_0-1} b_1^{e_1} \cdots b_m^{e_m}}(t) t^{-b_0} - \frac{\mathbf{Z}_X(t) t^{-b_0}}{\mathbf{Z}_X(t^{b_0})} \left[\text{Sym}^{e_0-1} X \times \prod_{i=1}^m \text{Sym}^{e_i} X \right].$$

(2)

$$\overline{K}_{X,1 \bullet b_0^{e_0} b_1^{e_1} \cdots b_m^{e_m}}(t) = t^{-\sum_i e_i b_i} \left(\mathbf{Z}_X(t) - \sum_{i=0}^m \frac{\mathbf{Z}_X(t)}{\mathbf{Z}_X(t^{b_i})} \prod_{l=i+1}^m [\text{Sym}^{e_l} X] t^{e_l b_l} \sum_{j=0}^{e_i-1} [\text{Sym}^j X] t^{j b_i} \right).$$

In the special cases when $m = 1$ or both $m = 2$ and $e_0 = 1$, Theorem 4.2 reduces to [VW12, Lemma 5.18, Proposition 5.19, Example 5.20]. Taking $\mathbf{k} = \mathbb{F}_q$ and applying the \mathbb{F}_q -point counting functor to Theorem 4.2 (2) with $X = \mathbb{A}^1$ gives Theorem 1.1 (using the basic fact that $\text{Sym}^a \mathbb{A}^1$ has q^a points for $a \geq 0$). As $[\text{Sym}^r \mathbb{A}^d] = [\mathbb{A}^{rd}]$ (e.g. see [Göt01, Lemma 4.4]), Theorem 4.2 (2) with $X = \mathbb{A}^d$ gives a very similar result to that of $X = \mathbb{A}^1$, just with each $[\mathbb{A}^s]$ replaced by $[\mathbb{A}^{sd}]$, and we have

$$(5) \quad \overline{w}_{1^k b_0^{e_0} \cdots b_m^{e_m}}(\mathbb{A}^d) = [\mathbb{A}^{d(k + \sum_i e_i)}]$$

for any $d \geq 0$. These are the cases that motivate Conjecture 1.2 (see [VW12, 1.41-1.44] for more details about this motivation).

Proof. The first part of (1) follows exactly as the same statement in the proof of Theorem 1.1. The second follows by multiplying both sides of the first by t^k , summing over k , and applying Lemma 3.2 (which has an analogous proof in the Grothendieck ring setting) and Lemma 4.1. Finally, (2) is proven inductively using Lemma 4.1 as a base case and the second part of (1) for the inductive step. \square

REFERENCES

- [Arn69] V. I. Arnol'd. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.
- [Arn70] V. I. Arnol'd. Certain topological invariants of algebraic functions. *Trudy Moskov. Mat. Obšč.*, 21:27–46, 1970.
- [Chu12] Thomas Church. Homological stability for configuration spaces of manifolds, 33 pages., arxiv:1103.2441, to appear in *Inventiones Mathematicae*, 2012.
- [Ell] Jordan Ellenberg. Motivic puzzle: the moduli space of squarefree polynomials. <http://quomodocumque.wordpress.com/2010/05/13/motivic-puzzle-the-moduli-space-of-squarefree-polynomials/>.
- [Göt01] Lothar Göttsche. On the motive of the Hilbert scheme of points on a surface. *Math. Res. Lett.*, 8(5-6):613–627, 2001.
- [Kap00] M. Kapranov. The elliptic curve in the s-duality theory and eisenstein series for kac-moody groups. Preprint, arXiv:math.AG/0001005, 2000.
- [McD75] Dusa McDuff. Configuration spaces of positive and negative particles. *Topology*, 14:91–107, 1975.
- [RW12] Oscar Randal-Williams. Homological stability for unordered configuration spaces. Preprint, <http://arxiv.org/abs/1105.5257>, to appear in *The Quarterly Journal of Mathematics*, 2012.
- [VW12] Ravi Vakil and Melanie Matchett Wood. Discriminants in the Grothendieck ring. 2012. arXiv:1208.3166.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, 480 LINCOLN DRIVE, MADISON, WI 53705 USA, AND AMERICAN INSTITUTE OF MATHEMATICS, 360 PORTAGE AVE, PALO ALTO, CA 94306-2244 USA

E-mail address: mmwood@math.wisc.edu